QN→ A Convergent sequence has a unique limit.

"or"

Prove that a convergent sequence determines its limit uniquely.

Ans.→ Let $\{a_n\}$ be a convergent sequence and it converges to l and also to l' where $l \neq l'$

Now, we have to show that it has got a unique limit.

To prove it, put.

$$\epsilon = \frac{1}{2}|l - l'|$$

As $\{a_n\}$ converges to l then there exist a no. p such that,

$$|a_n - l| < \epsilon \text{ for all } n \geq p.$$

Again, since $\{a_n\}$ also converges to l' then there exist a natural no. q such that,

$$|a_n - l'| \leq \epsilon \text{ for all } n \geq q$$

Put $r = \max\{p, q\}$ Then

$$|l - l'| = |l - a_r| + |a_r - l'|$$
$$\leq |l - a_r| + |a_r - l'|$$
$$< \epsilon + \epsilon = 2\epsilon = |l - l'|$$

i.e. $|l - l'| < |l - l'|$

which is absurd

Hence, $l = l'$

Thus the limit is unique.

QN.→ Prove that every Convergent sequence is bounded

or,

Every Convergent sequence of real nos. is a bounded sequence

Ans.→ Let $\{S_n\}$ be a Convergent sequence and it has a limit

Let, $\lim S_n = l$.

If $\epsilon = 1$, there exists $m \in N$ such that

$$|S_n - l| < 1 \; (n \geq m) \text{ --- (a)}$$

Now, $|S_n| = |S_n - l + l| \leq |S_n - l| + |l| < 1 + |l|$ — (b)

If, $K = \max\{|S_1|, |S_2|, \cdots, |S_{m-1}|\}$

then, $|S_n| < K + |l| + 1 \; (n \in N)$

Now puting $K + |l| + 1 = K_1$

then, $|S_n| < K_1$ for all $n \in N$.

∗ i.e. the sequence $\{S_n\}$ is bounded.

Q.No:- To State and Prove Abel's test.

Ans:- Statement:- If $\sum u_n$ converges and $a_1, a_2, a_3, \ldots$ is a sequence of positive terms, tending to finite limit, then $\sum a_n u_n$ is convergent.

Proof:- Since $\lim_{n\to\infty} a_n =$ a finite quantity, say l, it follows that

$$\lim (a_n - l) = 0.$$

Hence by Dirichlet's test $\sum (a_n - l) u_n$ is convergent

i.e. $\sum (a_n u_n - l u_n)$ is convergent. But $\sum u_n$ is convergent.

$\therefore \sum a_n u_n$ is also convergent.

—o—

Q.No:- To State and Prove Dirichlet's Test for the Convergence of series of arbitrary terms.

Ans:- Statement:- If for a series $\sum u_n$ the sequence $\{S_n\}$ such that $S_n = u_1 + u_2 + \cdots + u_n$ is bounded and if $\{v_n\}$ is a positive monotonically decreasing sequence tending to zero, then the series $\sum u_n v_n$ is convergent.

Proof:- We consider

$$\alpha_n = u_1 v_1 + u_2 v_2 + u_3 v_3 + \cdots + u_n v_n,$$

Then, $\alpha_n = u_1 v_1 + u_2 v_2 + u_3 v_3 + \cdots + u_n v_n$

$= S_1 v_1 + (S_2 - S_1) v_2 + (S_3 - S_2) v_3 + \cdots + (S_n - S_{n-1}) v_n$

$= S_1 (v_1 - v_2) + S_2 (v_2 - v_3) + \cdots + S_{n-1} (v_{n-1} - v_n) + S_n v_n$

$$= \sum_{k=1}^{n-1} S_k (v_k - v_{k+1}) + S_n v_n \quad \text{—— (1)}$$

Now, since the sequence $\{S_n\}$ is bounded and also the sequence $\{u_n\}$ is positive & monotonically decreasing, therefore by previous theorm

$\sum_{k=1}^{n-1} S_k (v_k - v_{k+1})$ tends to a finite lt as $n \to \infty$

Also, since $v_n \to 0$ as $n \to \infty$ and since $\{S_n\}$ is bounded, therefore $S_n v_n \to 0$ as $n \to \infty$. Hence from (1), we see that α_n tends to a finite limit as $n \to \infty$.

Hence, the series $\sum u_n v_n$ is convergent.